MULTIPHASE MODEL OF REINFORCED MATERIALS

Bruno Sudret[1] *and Patrick de Buhan*[2]
ENPC-CERCSO, 6-8 Av. Blaise Pascal, Cité Descartes, Champs-sur-Marne
F-77455 Marne-la-Vallée Cedex 2.

SUMMARY

Making use of the virtual work method, a mechanical multiphase model for reinforced materials is developed. Matrix and reinforcement are continuously distributed. The work of internal forces accounts for the matrix three-dimensional and inclusions one-dimensional characteristics and for their interaction as well. The corresponding equations of motion and associated boundary conditions are deduced for each phase by applying the virtual work principle. Two applications of the model are proposed in the domain of rock-bolting tunnels.

Keywords: Soil/rock reinforcement, virtual work method, multiphase model, computer simulation.

1. INTRODUCTION

Reinforcement techniques are increasingly used in different areas of geotechnical engineering (rock bolting for tunnels, soil nailing for retaining structures. "micropiling" techniques, ...). In such structures, inclusions are periodically disposed along one or two directions. The density of inclusions makes it impossible to treat them separately. From a macroscopical viewpoint, the *composite* made of soil and inclusions can be replaced by a homogeneous material, whose mechanical properties are deduced from those of the constituents.

This equivalent material is modeled as a *multiphase reinforced* continuum by the *method of virtual work* developed by Germain (1986) and Salençon (1996). This method, which constitutes an alternative formulation to the classical principle of momentum balance to derive the equations of motion, will be shortly described in the sequel and then applied step-by-step to model the reinforced material.

First of all, one defines the geometry of the described system S and its subsystems S'. One then chooses the virtual velocity fields denoted by $\hat{U}$, which form a vectorial space. This space contains in particular the rigid body virtual motions (r.v.m) and the allowed real motions of the system. For each subsystem, the appropriate expressions for the

---

[1] PhD student
[2] Professor in Structural Engineering

virtual work of internal forces $P_{(i)}$, external forces $P_{(e)}$ and inertial forces $\mathcal{A}'$ are then postulated. The *principle of virtual work* (P.V.W) then writes:

$$\forall\, S' \subset S, \qquad \forall\, \hat{\mathrm{U}} \text{ r.v.m}, \quad P_{(i)}(\hat{\mathrm{U}}) = 0 \tag{1-a}$$

$$\forall\, \hat{\mathrm{U}} \text{ v.m}, \quad P_{(i)}(\hat{\mathrm{U}}) + P_{(e)}(\hat{\mathrm{U}}) = \mathcal{A}'(\hat{\mathrm{U}}) \tag{1-b}$$

2. MULTIPHASE MODEL OF REINFORCED MATERIAL

2.1 Geometrical description

Let us consider a volume Ω of a three-dimensional medium reinforced by a network of uniformly distributed inclusions parallel to N distinct directions characterized by unit vectors e_r, $r = 1, .. N$. One attaches at each geometric point $\underline{x}$ of Ω, a *matrix* particle and N *reinforcement* particles to form by superposition *the* particle of reinforced material. This allows to define a *phase* as the set of particles being of the same type, a *single-phase system* S'^j as the set of the phase-j particles, $j \in \{m;\ r = 1, ..\ N\}$ included in a volume $\Omega' \subset \Omega$ and *a multiphase system* S' as the set of all the particles included in Ω'. The virtual motions are given by $(N + 1)$ continuous and differentiable velocity fields denoted by $\underline{\hat{U}}^{\,m}$ for the matrix phase and $\underline{\hat{U}}^{\,r}$, $r = 1, ..\ N$ for the reinforcement phases:

$$\hat{\mathrm{U}}(\underline{x}) = \{\underline{\hat{U}}^{\,m}(\underline{x});\quad \underline{\hat{U}}^{\,r}(\underline{x});\quad r = 1, ..\ N\} \tag{2}$$

2.2 Expression for the virtual work

2.2.1 Internal forces

For each given subsystem filling a volume Ω', the work of internal forces is supposed to be obtained by the integration of a volume density. This density is made of terms related to each phase and of *interaction terms* between matrix and reinforcement phases. No interaction between reinforcement phases themselves is accounted for.

The matrix phase is modeled as a continuum, so that the work density of internal forces is given as the following linear combination of the velocity field and its first derivatives:

$$p'^{m}_{(i)}\left(\hat{\mathrm{U}}(\underline{x})\right) = -\left(\underline{A}^{m}(\underline{x})\cdot\underline{\hat{U}}^{\,m}(\underline{x}) + \underline{\underline{s}}^{\,m} : \underline{\underline{\mathrm{grad}\hat{U}}}^{\,m}(\underline{x})\right) \tag{3}$$

The reinforcements are modeled as *uniformly distributed bars* transmissing only tensile-compressive forces. The corresponding work density of internal forces then writes:

$$p'^{r}_{(i)}\left(\hat{\mathrm{U}}(\underline{x})\right) = -\left(\underline{A}^{r}(\underline{x})\cdot\underline{\hat{U}}^{\,r}(\underline{x}) + \underline{\underline{s}}^{\,r}(x)\cdot\frac{d\underline{\hat{U}}^{\,r}(\underline{x})}{ds^{r}}\right) = -\left(\underline{A}^{r}(\underline{x})\cdot\underline{\hat{U}}^{\,r}(\underline{x}) + \left(\underline{e}_r \otimes \underline{\underline{s}}^{\,r}(\underline{x})\right) : \underline{\underline{\mathrm{grad}\hat{U}}}^{\,m}(\underline{x})\right)$$

$\underline{s}_r$ being the abscissa along the $\underline{e}_r$ direction. (4)

The interaction between phases is supposed to be *pinpoint*-like. Thus the corresponding expression is:

$$p'^{I}_{(i)}\left(\hat{U}(\underline{x})\right) = -\left(\underline{I}^{m}(\underline{x})\cdot\underline{\hat{U}}^{m}(\underline{x}) + \sum_{r=1}^{N}\underline{I}^{r}\cdot\underline{\hat{U}}^{r}(\underline{x})\right) \tag{5}$$

The global expression for the work of internal forces depends on the single- or multiphase characteristic of the current subsystem. In the first case, one writes $P'_{i}\left(\hat{U}\right) = \int_{\Omega'} p'^{j}_{(i)}\left(\hat{U}(\underline{x})\right) d\Omega'$. In the second case, the contributions of each phase as well as the interaction terms have to be taken into account.

2.2.2 External forces

For a single-phase subsystem S'^{j} filling a volume Ω', one considers three different types of external forces:

- volume forces $r^{j}(\underline{x})\underline{F}^{j}(\underline{x})$ (gravity,..) applied by the outside of the whole system,
- surface forces applied on the boundary ∂Ω' by the particles of phase *j* being outside Ω', and denoted by $\underline{T}^{j}_{\Omega'}(\underline{x})$,
- interaction forces with the other phases, given by one (resp. *N*) volume density(ies) if $j = r$ (resp. $j = m$).

The virtual work of external forces for a single-phase subsystem then writes:

$$P'^{j}_{(e)}\left(\hat{U}\right) = \int_{\Omega'} r^{j}(\underline{x})\underline{F}^{j}(\underline{x})\cdot\underline{\hat{U}}^{j}(\underline{x})\, d\Omega' + \int_{\partial\Omega'}\underline{T}^{j}_{\Omega'}(\underline{x})\cdot\underline{\hat{U}}^{j}(\underline{x})\, dS' - \int_{\Omega'}\underline{I}^{j}(\underline{x})\cdot\underline{\hat{U}}^{j}(\underline{x})\, d\Omega' \tag{6}$$

For a multiphase subsystem, the interaction forces become *internal* forces, which are described above. The virtual work of external forces reduces to:

$$\begin{aligned} P'^{j}_{(e)}\left(\hat{U}\right) = & \int_{\Omega'}\left(r^{m}(\underline{x})\underline{F}^{m}(\underline{x})\cdot\underline{\hat{U}}^{m}(\underline{x}) + \sum_{r=1}^{N} r^{r}(\underline{x})\underline{F}^{r}(\underline{x})\cdot\underline{\hat{U}}^{r}(\underline{x})\right) d\Omega' \\ & + \int_{\partial\Omega'}\left(\underline{T}^{m}_{\Omega'}(\underline{x})\cdot\underline{\hat{U}}^{m}(\underline{x}) + \sum_{r=1}^{N}\underline{T}^{r}_{\Omega'}(\underline{x})\cdot\underline{\hat{U}}^{r}(\underline{x})\right) dS' \end{aligned} \tag{7}$$

2.2.3 Inertial forces

The following classical expression is chosen to represent the virtual work of inertial forces:

$$\mathcal{A}'(\hat{U}) = \int_{\Omega'}\left(r^{m}(\underline{x})\underline{g}^{m}(\underline{x})\cdot\underline{\hat{U}}^{m}(\underline{x}) + \sum_{r=1}^{N} r^{r}(\underline{x})\underline{g}^{r}(\underline{x})\cdot\underline{\hat{U}}^{r}(\underline{x})\right) d\Omega' \tag{8}$$

where $\underline{g}^{j}(\underline{x})$ denotes the acceleration of the phase-*j* particle at point $\underline{x}$.

2.3 Application of the P.V.W

2.3.1 First statement

The first statement of the P.V.W (1-a) requires the virtual work of internal forces to vanish for all rigid body virtual motions. By introducing a translation velocity field for a single-phase subsystem S'^j, $j \in \{m;\ r = 1, .. N\}$: $\hat{U}(\underline{x}) = \{0;\ \ ,\underline{\hat{U}}_0^j,0,\ldots\}$, one proves that $\underline{A}^m(\underline{x}) = \underline{A}^r(\underline{x}) = \underline{0}$. By choosing a rotation motion field, one obtains the symmetry of the tensors $\underline{\underline{s}}^m(\underline{x})$ and $\underline{e}_r \otimes \underline{s}^r(\underline{x})$, the second condition being equivalent to $\underline{s}^r(\underline{x}) = s^r(\underline{x})\underline{e}_r$. Then by choosing a rigid body virtual motion for a multiphase subsystem, one can rewrite the interaction term (4) as:

$$p'^{I}_{(i)}(\hat{U}) = -\sum_{r=1}^{N} \underline{I}^r(\underline{x})\cdot\left(\underline{\hat{U}}^r(\underline{x}) - \underline{\hat{U}}^m(\underline{x})\right) \tag{9}$$

in which the reinforcement-to-matrix *relative* velocities appear. Finally the work of internal forces becomes:

$$P_{(i)}(\hat{U}) = -\int_{\Omega'}\left\{\underline{\underline{s}}^m(\underline{x}) : \underline{\underline{\mathrm{grad}\hat{U}}}^m(\underline{x}) + \sum_{r=1}^{N}\left(s^r(\underline{x})\underline{e}_r \otimes \underline{e}_r\right) : \underline{\underline{\mathrm{grad}\hat{U}}}^r(\underline{x})\right\} d\Omega' \tag{10}$$

$$-\int_{\Omega'}\sum_{r=1}^{N}\underline{I}^r(\underline{x})\cdot\left(\underline{\hat{U}}^r(\underline{x}) - \underline{\hat{U}}^m(\underline{x})\right) d\Omega'$$

2.3.2 Second Statement

By substituting Eqs.(7), (8) and (10) in Eq.(1-b), one can factorize the terms associated with each velocity field $\underline{\hat{U}}_0^j$, $j \in \{m;\ r = 1, .. N\}$, provided that the divergence theorem is applied to the gradient terms. The (N+ 1) following equations of motion are then derived from a classical reasoning:

$$\mathrm{div}\,\underline{\underline{s}}^m(\underline{x}) + r^m(\underline{x})\left(\underline{F}^m(\underline{x}) - \underline{g}^m(\underline{x})\right) + \sum_{r=1}^{N}\underline{I}^r(\underline{x}) = \underline{0} \tag{11-a}$$

$$\mathrm{div}\left(s^r(\underline{x})\underline{e}_r \otimes \underline{e}_r\right) + r^r(\underline{x})\left(\underline{F}^r(\underline{x}) - \underline{g}^r(\underline{x})\right) - \underline{I}^r(\underline{x}) = \underline{0},\ r = 1, .. N \tag{11-b}$$

Eq.(11-b) rewrites simply $\dfrac{ds^r(\underline{x})}{ds^r}\underline{e}_r + r^r(\underline{x})\left(\underline{F}^r(\underline{x}) - \underline{g}^r(\underline{x})\right) - \underline{I}^r(\underline{x}) = \underline{0}$. The boundary conditions are also deduced:

$$\underline{T}^m_{\Omega'}(\underline{x}) = \underline{\underline{s}}^m(\underline{x})\cdot\underline{n}(\underline{x}) \tag{12-a}$$

$$\underline{T}^r_{\Omega'}(\underline{x}) = s^r(\underline{x})\left(\underline{n}(\underline{x})\cdot\underline{e}_r\right)\underline{e}_r,\ r = 1, .. N \tag{12-b}$$

The internal forces finally prove to be described by a stress tensor $\underline{\underline{s}}^m$ in the matrix, a scalar stress s^r for each reinforcement phase, and N interaction fields $\underline{I}^r$.

2.3.3 Perfect bonding hypothesis

In most practical cases in the area of civil engineering, there is no possibility of slippage between the reinforcing inclusions and the continuum. In the present multiphase model, this *perfect boding hypothesis* is accounted for by prescribing the equality of the velocity fields in the different phases. Thus the interaction terms vanish. To eliminate them from the above equations, it is convenient to sum up Eqs.(11-12) over all phases. This yields:

$$\forall\,\Omega' \subset \Omega,\quad \begin{cases} \mathrm{div}\,\underline{\underline{\Sigma}}(\underline{x}) + r(\underline{x})\left(\underline{F}(\underline{x}) - \underline{g}(\underline{x})\right) = \underline{0} & \forall \underline{x} \in \Omega' \\ \underline{T}_{\Omega'}(\underline{x}) = \underline{\underline{\Sigma}}(\underline{x})\cdot\underline{n}(\underline{x}) & \forall \underline{x} \in \partial\Omega' \end{cases} \tag{13}$$

where:

$$\rho \underline{F} = \sum_j \rho^j \underline{F}^j \,, \qquad \underline{T}_{\Omega'} = \sum_j \underline{T}^j_{\Omega'} \,, \qquad \underline{\underline{\Sigma}} = \underline{\underline{\sigma}}^m + \sum_{r=1}^{N} \sigma^r \underline{e}_r \otimes \underline{e}_r \qquad \rho \underline{\gamma} = \sum_j \rho^j \underline{\gamma}^j \,. \tag{14}$$

Eqs.(13) appears to be the classical equations of motion and boundary conditions of the single-phase Cauchy continuum. $\underline{\underline{\Sigma}}$ represents the *global stress tensor* whereas $\underline{\underline{\sigma}}^m$ (resp. σ^r) denotes the *partial* stresses in the matrix (resp. the reinforcement) phase.

2.4 Elastic constitutive law

In the framework of *small perturbations* and perfect bonding hypothesis, it is possible to derive a global constitutive law for the reinforced material from the characteristics of each constituent. As a matter of fact, each phase has the same displacement field, namely $\underline{\xi}(\underline{x})$. It is then possible to compute *a global strain tensor* $\underline{\underline{\epsilon}}$ and to relate it to the partial strain variables as follows:

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left(\underline{\underline{\mathrm{grad}}}\,\underline{\xi} + {}^t\underline{\underline{\mathrm{grad}}}\,\underline{\xi}\right) = \underline{\underline{\epsilon}}^m \tag{15-a}$$

$$\epsilon^r = \frac{\partial \xi_r}{\partial s_r} = \underline{\underline{\epsilon}} : \left(\underline{e}_r \otimes \underline{e}_r\right) \quad r = 1, \ldots N \tag{15-b}$$

Let us assume now the following elastic constitutive laws for each individual phase:

$$\underline{\underline{\sigma}}^m = \underline{\underline{\underline{\underline{a}}}}^m : \underline{\underline{\epsilon}}^m \,; \qquad \sigma^r = a^r \epsilon^r \,, \qquad r = 1, \ldots N \tag{16}$$

where $\underline{\underline{\underline{\underline{a}}}}^m$ is the tensor of elastic moduli of the matrix and a^r the reinforcement stiffness. By combining Eqs.(15) and (16), one obtains the *tensor of global elastic moduli* $\underline{\underline{\underline{\underline{A}}}}^m$ relating global stress and strain tensors:

$$\underline{\underline{\Sigma}} = \underline{\underline{\underline{\underline{A}}}} : \underline{\underline{\epsilon}} \,, \quad \underline{\underline{\underline{\underline{A}}}} = \underline{\underline{\underline{\underline{a}}}}^m + \sum_{r=1}^{N} a^r \underline{e}_r \otimes \underline{e}_r \otimes \underline{e}_r \otimes \underline{e}_r \tag{17}$$

The latter equation proves the anisotropic behaviour of the reinforced material even if the matrix alone is isotropic.

3. APPLICATIONS

The stability of tunnels excavated in rockmasses remains a major issue for geotechnical engineers. In order to reduce the radial displacement field at the tunnel wall *(convergence)* and to improve the global stability of the structure, the use of metallic inclusions (bolts) has increased in the last few years. The bolt density in such usual structures and the use of grouting techniques allow to treat these inclusion reinforced materials according to the multiphase model presented in section 2.

Two different applications are presented in the sequel. At first one develops all the constitutive equations for the general model by introducing in particular a linear constitutive law for the interaction forces. This leads to a system of ordinary differential equations, which is solved numerically by the finite difference method. In a second part, one considers the perfect bonding hypothesis. The problem is solved analytically and by the finite element method. Results are compared with one another.

3.1 Problem statement

Let us consider a horizontal cylindrical tunnel of radius R excavated in a homogeneous rockmass, which was initially submitted to a hydrostatic stress field $\underline{\underline{s}}_0 = -P_0 \underline{\underline{1}}$, $P_0 = gH$, where g is the specific weight of the rock. After a certain delay, bolts are placed radially into the rockmass (angular spacing a, horizontal spacing p), but we disregard here any intermediate construction phase. Thus the problem under consideration is that of a cylinder in a prestressed rock-bolted medium, at the wall of which a null pressure is applied. Cylindrical coordinates (r, q, z) are used in the sequel.

According to the results of Greuell (1993) and Bernaud *et al.* (1995), the influence of the bolt length l_b on the solution is negligible as soon as $l_b > 2R$. Thus for the sake of simplicity, we consider here an infinitely bolted rockmass. According to the notations in Fig. 1, the bolt density at the tunnel wall is given by $d_b = 1/(a\, pR)$. Considering a representative volume of the periodic reinforcement scheme (Fig. 1-b), one can determine the volume fraction of reinforcement at each point $\underline{x}$ of the massif by:

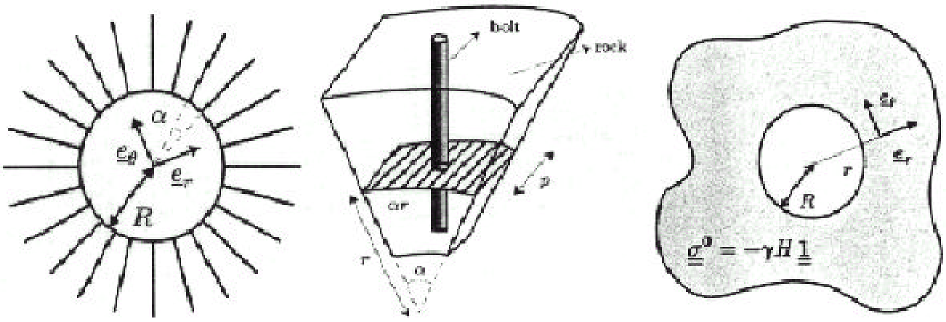
$$h(\underline{x}) = \frac{S_b}{a\ pr} = d_b S_b \frac{R}{r} \tag{18}$$

3.2 Parameters for the multiphase model

The rockmass is supposed to obey a homogeneous isotropic constitutive law, represented by its Lamé's coefficients l and m, or equivalently its Young modulus E^r and Poisson's ration v^r. The reinforcement stiffness a_r to be introduced accounts for uniformly distributed bolts. Due to the symmetry, it is evaluated by:

$$a_r(r) \equiv K(r) = h(r) E_b \tag{19}$$

where E_b is the Young modulus of the bolt constitutive material (steel). It yields finally $K(r) = K_0 / r$, with $K_0 = d_b\, S_b\, E_b\, R$. It clearly appears that the reinforcement fraction decreases with the radius.



a – Original reinforcing scheme *b – Representative volume of bolted rockmass* *c – Multiphase model*

Figure 1: Geometrical description of the problem

3.3 Semi-analytical solution for the non adherent model

3.3.1 Constitutive equations

Let us first consider the general multiphase model in which the rockmass (matrix) and the reinforcements (bolts) have different kinematics. Due to the symmetry of the problem, the displacement fields are of the form:

$$\underline{x}^m(\underline{x}) = M(r)\underline{e}_r \ ; \ \underline{x}^r(\underline{x}) = B(r)\underline{e}_r \tag{20}$$

The associated strain variables are given by:

$$\underline{\underline{e}}^m(\underline{x}) = M'(r)\underline{e}_r \otimes \underline{e}_r + \frac{M(r)}{r}\underline{e}_q \otimes \underline{e}_q \ ; \ e^r(\underline{x}) = B'(r) \tag{21}$$

The respective constitutive laws (16) write here:

$$\underline{\underline{s}}^m = \underline{\underline{s}}^0 + l(\mathrm{tr}\underline{\underline{e}}^m) + 2m\underline{\underline{e}}^m \tag{22-a}$$

$$s^r = K(r)e^r \tag{22-b}$$

We complete the system of equations by introducing a linear constitutive law for the interaction forces related to the relative displacement field:

$$\underline{I}(\underline{x}) = a\left[B(r) - M(r)\right]\underline{e}_r \tag{23}$$

The equations of motion (11) reduce here to:

$$\mathrm{div}\,\underline{\underline{s}}^m + \underline{I} = \underline{0} \tag{24-a}$$

$$\mathrm{div}\left(s^r \underline{e}_r \otimes \underline{e}_r\right) - \underline{I} = \underline{0} \tag{24-b}$$

The substitution of Eqs. (22 – 23) in (24) leads to the following differential system:

$$\begin{cases} \dfrac{K_o}{r}B''(r) - a\left[B(r) - M(r)\right] = 0 \\ (l + 2m)\left(M'(r) + \dfrac{M(r)}{r}\right)' + a\left[B(r) - M(r)\right] = 0 \end{cases} \tag{25}$$

3.3.2 Boundary conditions

We impose that the displacement fields vanish at infinity and that the tunnel wall is stress-free:

$$\lim_{r \to +\infty} B(r) = \lim_{r \to +\infty} M(r) = 0 \tag{26-a}$$

$$s^r(r = R) = \frac{K_0}{R}B'(R) = 0 \tag{26-b}$$

$$s^m_{rr}(r = R) = (l + 2m)M'(R) + l\,\frac{M(R)}{R} - P_0 = 0 \tag{26-c}$$

The boundary value problem given by Eqs.(25) and (26) is numerically solved by a finite difference algorithm. The displacement fields are given in Fig. 2. It clearly appears that both fields are significantly different only in the vicinity of the tunnel wall. The relative displacement *B(r) - M(r)* is at first positive and becomes negative. This shows a two-zone mechanism: the bolts first retain the rock for small values of *r*, and afterwards the bolts are anchored in the rock. This perfectly corroborates the observations of the shear forces made in soil-nailing experiences.

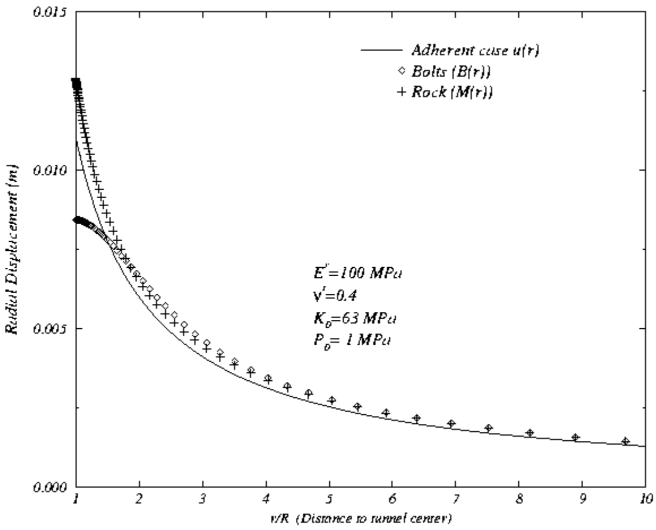


Figure 2: Displacement curves

3.4 Analytical solution for the adherent model

The amplitude of the anchor zone depends on the interaction parameter a. The case of $a \to \infty$ corresponds to the perfect bonding hypothesis. In this limit case, the simplified formalism involving global strain and stress tensors can be then adopted, and the problem completely analytically solved. Details are given in Greuell (1993). The equation governing the radial displacement *u(r)* writes:

$$\left(1+k\frac{R}{r}\right)u''(r)+\frac{u'(r)}{r}-\frac{u(r)}{r^2}=0 \ \text{ with } \ k=\frac{K_0/R}{l+2m} \tag{27}$$

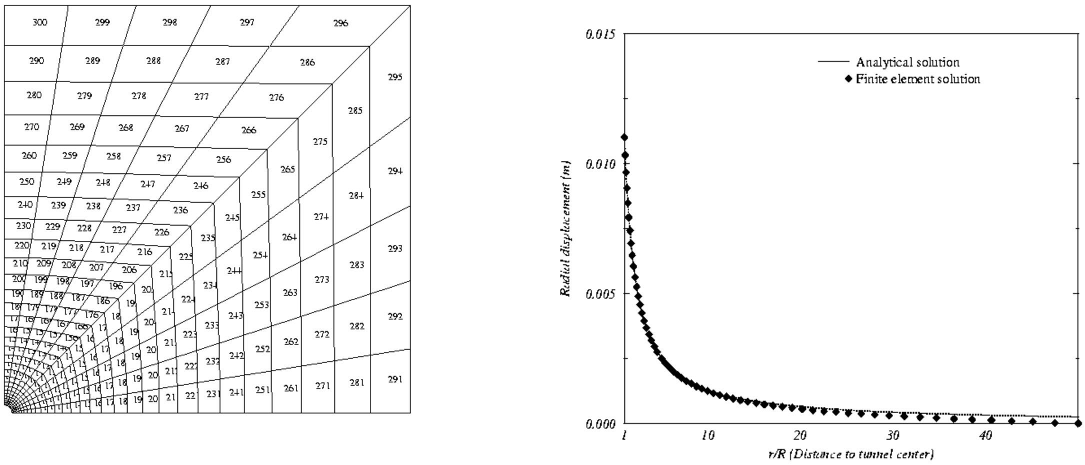
Its analytical solution with the same boundary conditions as (26) writes:

$$u(r)=\frac{-P_0 R}{(l+2m)\ln(1+k)-2(l+m)(1-\ln(1+k)/k)}\left(1-\frac{r}{kR}\ln\left(1+\frac{kR}{r}\right)\right) \tag{28}$$

The corresponding curve has been plotted in Fig. 2 as well. It proves to be asymptotically tangent to the curves determined in the framework of section 3.3.

3.5 Finite Element implementation

Finally the multiphase adherent problem is solved by a numerical procedure implemented in a finite element code. The main point is that the mesh to be used (Fig. 3-a) has the same density all over the reinforced zone, since the inclusions have not to be treated separately from the rockmass. The radial displacement drawn in Fig. 3-b shows a perfect agreement with the analytical result given by Eq.(28), the only discrepancy being caused by the finite dimensions of the mesh.



a – FE mesh *b – Radial displacement*

Figure 3: Comparison between analytical and numerical results (perfect bonding hypothesis)

4. CONCLUSION

The model presented in this paper allows analytical solutions for problems dealing with reinforced media, as well as their numerical treatment by the finite element method. Only the elastic behaviour has been treated here, but it is possible to derive a complete elastic plastic constitutive law (for details, see de Buhan et Sudret (1998)).

The virtual work method allows an easy construction of the equations of motion as shown in section 2. It is furthermore quite easy to enrich the model in order to account for the bending and shear forces in the reinforcement. The obtained model is a Cosserat continuum with identical properties as that obtained by de Buhan *et al.* (1998) by homogenization techniques.

5. REFERENCES

Bernaud,D., de Buhan, P. and Maghous, S. (1995) "Numerical simulation of the convergence of a bolt-supported tunnel through a homogenization method", *Int. J. Num. Met. Geo.*, vol. 19, pp 267-288.

de Buhan, P. and Dormieux, L. and Salençon, J. (1998) "Yield design of inclusion-reinforced materials as micropolar continua", *C.R.Ac.Sc., Paris* , Vol. 326 II-b, pp 163-170.

de Buhan, P. and Sudret, B. (1998) "A two-phase elastoplastic model for unidirectionally reinforced materials and its numerical implementation", *Int. J. Plas.*, submitted for publication.

Germain, P.(1986) *Mécanique, I et II*, Ellipses, Paris.

Greuell, E.(1993) *Etude du soutènement des tunnels Par boulons Passifs dans les sols et les roches tendres, par une méthode d'hornogénéisation*, Ph.D. thesis, Ecole Polytechnique. Salençon, J.(1996) *Mécarcique des milieux corctinus, tome I. Concepts généraux*, Ellipses. AUPELF/UREF, Paris.