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BUCKLING OF BEAMS WITH LOW SHEAR STIFFNESS

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SUMMARY

In the engineering practice it is very important that the structures have an adequate safety against the loss of stability. One form of the loss of stability is the lateral torsional buckling of beams. Solutions were worked out in the past to determine the critical load of a simple beam. There are structures where the above mentioned formulas cannot be applied e.g. beams with openings and sandwich structures where large shear deformations arise. In the following paper the authors will present two methods - that will take into account the effect of the shear deformation - to determine the critical load for these types of beams. At the end a short and simple solution is presented.

Keywords: buckling, beams, shear deformation

I. INTRODUCTION

Beams subjected to bending moments (and/or vertical loads) may loss their stability by lateral torsional buckling. If the applied load reaches a certain value (the critical load), the beam suddenly moves perpendicular to the plane of the applied load. This phenomenon, called lateral torsional buckling, plays a very important role in the design of slender beams, especially if the twist of the ends of the beams are not restrained, e.g. in the case of moving and lifting of beams.

Beams are commonly manufactured with openings where the shear deformation is significant. Sometimes trusses or frame structures are applied as beams, and for those structures, the shear deformation may be higher than the bending deformation. Composite beams and sandwiches also show high shear deformation.

Closed form solutions and simple formulas were determined in the past to calculate the critical load for beams with symmetrical cross-section taking into account the bending stiffnesses and the torsional stiffnesses of the beam. The authors are not aware of any closed form solution, which takes into account the effect of shear deformation. By neglecting the shear deformation the buckling load of a beam may be overestimated.

The goal of this paper is to find a method to calculate the critical load of beams taking into account the shear deformation.

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2. BUCKLING ANALISYS USING THE EQUILIBRIUM METHOD

Consider a symmetric, prismatic beam of length *l* (*Fig 1*). The beam is simply supported at both ends. The vertical displacement is restricted but the cross-section may rotate around the *x* and *y*-axes. The twist of the ends of the beam i.e. the rotation of the cross-sections around the *z*-axis is also restricted. The beam is subjected to a uniformly distributed load (*q*) in its symmetry plane (*Fig 1*).

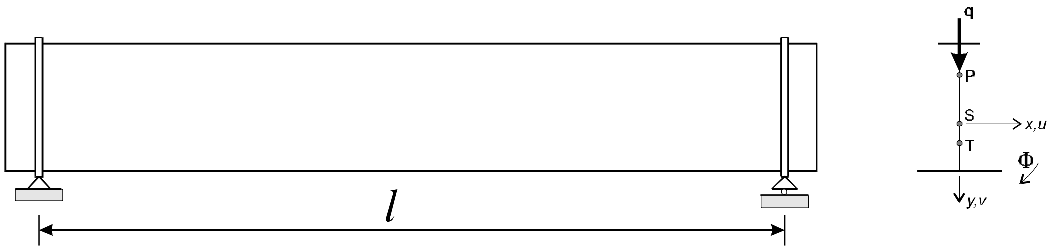

Fig. 1

In *Fig. 1* point *T* is indicating the shear center of the cross-section, point *S* is the center of gravity, and point *P* indicates the horizontal line along the vertically distributed load (*q*) acts.

The goal is to determine the critical load intensity (q_{cr}) causing the buckling of the beam.

The beam may twist about its axis (Φ), may move vertically (*v*), and may move horizontally (*u*). The horizontal displacement of a beam can be put together from two parts, the bending and the shear displacements:

$$u_T = u_D + u_S \tag{2-1}$$

where

u_T is the total horizontal (*x* directional) displacement of the shear center,

u_D is the horizontal displacement from bending,

u_S is the horizontal displacement from the shear deformation.

Between the two displacements the following connection can be made:

$$u_S = -\frac{D_y}{S} u''_D, \tag{2-2}$$

where

D_y is the bending stiffness of the beam in the *y* direction,

S is the shear stiffness in the horizontal plane.

The equilibrium equations of a basic beam element are:

$$D_y u_D^{IV} = \overline{q}_x \tag{2-3a}$$

$$D_\omega \Phi^{IV} - D_t \Phi'' = \overline{m}_{zT} \tag{2-3b}$$

where

D_y	is the bending stiffness of the beam in the *y* direction,
D_w	is the warping stiffness of the beam,
D_t	is the torsional stiffness of the beam,
u_D	is the displacement of the shear center in the *x* direction from bending,
Φ	is the twisting of the sections,
$\overline{q}_x$	is the load in *x* direction,
$\overline{m}_{zT}$	is the torque in the shear center.

These equilibrium equations are almost identical with the equations can be found in the literature. The only difference is that Eq. (2-3a) contains only the horizontal displacement from bending (u_D).

The detailed determination of the $\overline{q}_x$ and $\overline{m}_{zT}$ loading parts can be found in the literature so only the final results are presented:

$$D_y u_D^{IV} + (M_x \Phi)'' + N(u_T'' + y_o \Phi'') = 0, \qquad \text{(2-4a)}$$

$$D_\omega \Phi^{IV} - D_t \Phi'' + M_x u_T'' + t M_x'' \Phi - \beta_1 (M_x \Phi')' + N(y_o u_T'' + i_{pT}^2 \Phi'') = 0 \qquad \text{(2-4b)}$$

where

M_x	is the bending moment from *q* vertical load,
y_o	is the distance between the center of gravity and the shear center,
b_1	is a geometrical value that describes the section:

$$b_1 = J_1 + J_2 - 2y_o$$

$$J_1 = \frac{\int_{(A)} y^3 \mathrm{d}A}{I_x}, \quad J_2 = \frac{\int_{(A)} x^2 y \mathrm{d}A}{I_x},$$

N	is the axial force of the beam
i_{pT}	is the inertia circle of the section on the *T* shear center:

$$i_{pT}^2 = i_x^2 + i_y^2 + y_o^2.$$

The (2-4a) and (2-4b) equilibrium equations can be found in the literature. The only difference is that the first member (2-4a) contains the horizontal displacement from bending (u_D) not the total horizontal displacement (u_T). The above mentioned two equations contain three unknown functions: u_D, u_S and Φ. For the description of the problem (2-1) and (2-2) equations must be considered.

2.1 Solution with constant bending moments at both ends

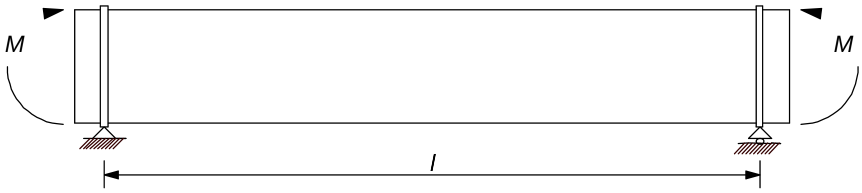

Fig. 2

In case of gable-like supported beam loaded with constant bending moments at both ends (*Fig.3*) the simplified form of the differential equations is the following:

$$D_y u_D^{IV} + M_x \Phi'' = 0\,, \tag{2-5a}$$

$$D_w \Phi^{IV} - D_t \Phi'' + M_x u_T'' - b_1 M_x \Phi'' = 0\,, \tag{2-5b}$$

$$u_T = u_D + u_S\,, \tag{2-5c}$$

$$u_S = -\frac{D_y}{S} u_D''\,. \tag{2-5d}$$

We are searching for the solution in the following form:

$$u_D = U_1 \sin\frac{p}{l} z \tag{2-6a}$$

$$u_S = U_1 \frac{p^2}{l^2}\frac{D_y}{S}\sin\frac{p}{l} z \tag{2-6b}$$

$$\Phi = \Phi_1 \sin\frac{p}{l} z \tag{2-6c}$$

With the application of these functions the boundary conditions of the gable-like support are satisfied in the z=0 and z=l sections.

The above-applied displacement functions (u_D and u_S) also satisfy Eq. (2-5d). By substituting the solution functions into Eq. (2-5a-c) the following algebraic system of equations arises:

$$\begin{bmatrix} D_y \dfrac{p^2}{l^2} & -M_x \\ -M_x\left(1+\dfrac{p^2}{l^2}\dfrac{D_y}{S}\right) & D_w \dfrac{p^2}{l^2} + D_t + b_1 M_x \end{bmatrix} \begin{bmatrix} U_1 \\ \Phi_1 \end{bmatrix} = 0 \tag{2-7}$$

The condition of the buckling is the non-trivial solution: the determinant of the coefficient matrix must be zero.

$$\begin{vmatrix} D_y \dfrac{p^2}{l^2} & -M_x \\ -M_x\left(1+\dfrac{p^2}{l^2}\dfrac{D_y}{S}\right) & D_w \dfrac{p^2}{l^2} + D_t + b_1 M_x \end{vmatrix} = 0 \tag{2-8}$$

From Eq. (2-8) we become the following quadratic equation:

$$\frac{1}{\overline{D}_y} M_{cr}^2 - \frac{\pi^2}{l^2} \beta_1 M_{cr} - \frac{\pi^2}{l^2} \left(D_t + \frac{\pi^2}{l^2} D_\omega \right) = 0 \tag{2-9}$$

where

$$\overline{D}_y = \frac{1}{\dfrac{1}{D_y} + \dfrac{\pi^2}{l^2} \dfrac{1}{S}} \tag{2-10}$$

Eq. (2-9) is identical with the solution that can be found in the literature. But instead of the bending stiffness (D_y) we apply a “modified bending stiffness” that contains the effect of the shear deformation.

3. DETERMINATION OF THE CRITICAL LOAD WITH ENERGY METHOD

In the following we will determine the critical load of a lifted beam loaded with q uniformly distributed load (*Fig. 3*).

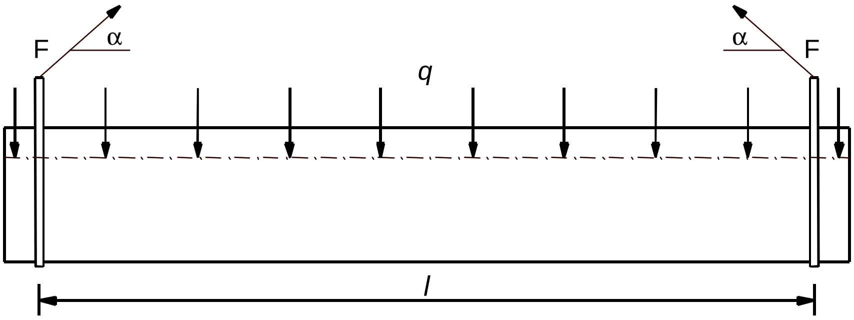

Fig.3.

The basic equation gives the equilibrium of external and internal works:

$$L_{ext} - L_{\text{int}} = 0 \tag{3-1}$$

The internal and external works are dependent from the displacements, which are presumed as series of functions. The unknown coefficients ($c_1, c_2, \ldots, c_n$) can be determined from the following conditions

$$\frac{\partial \Pi}{\partial c_1} = 0 \qquad i=1,2,\ldots,n.$$

The equation of the internal work:

$$L_{\text{int}} = \frac{D_t}{2} \int_0^l (\Phi')^2 \mathrm{dz} + \frac{D_\omega}{2} \int_0^l (\Phi'')^2 \mathrm{dz} + \frac{D_y}{2} \int_0^l (u_D'')^2 \mathrm{dz} + \frac{S}{2} \int_0^l (u_S')^2 \mathrm{dz} \tag{3-2}$$

where the last term contains the inner work from the shear deformation.

The expression of external work can be assembled from five parts:

a) The uniformly distributed load acts in the shear center of the beam. During the depression of the load external work arises.

b) The vertical load acts in the real point of its application. This point twists around the shear center and sinks.

c) We release the fix supports and allow the slew of twisted beam as a rigid body.

d) The horizontal force-components of the slanted cables cause compression. Because of the deformation mentioned in point a) these two forces come closer to each other.

e) The point of suspension is usually not consistent with the center of gravity. The horizontal force mentioned in the previous point acts non-axial.

In the above mentioned five cases the external works are the following:

$$\text{a)} \quad L_{ext}^{gable} = \frac{1}{2}\int_0^l \left[-b_1 M_x (\Phi')^2 - 2M_x \Phi u_T''\right]\mathrm{dz} \tag{3-3a}$$

$$\text{b)} \quad L_{ext}^{eccentr} = \frac{t}{2} q \int_0^l \Phi^2 \mathrm{dz} \tag{3-3b}$$

$$\text{c)} \quad L_{ext}^{susp} = \frac{q}{2\,fl}\left[\int_0^l (u_T + t\Phi)\mathrm{dz}\right]^2 \tag{3-3c}$$

$$\text{d)} \quad L_{ext}^{(N)} = \frac{ql}{4}\cot\alpha \int_0^l [u_T' + (t+f)\Phi']^2\,\mathrm{dz} \tag{3-3d}$$

$$\text{e)} \quad L_{ext}^{(M_N)} = a\frac{ql}{4}\cot\alpha \int_0^l \left[-\beta_1 (\Phi')^2 - 2\Phi u_T''\right]\mathrm{dz} \tag{3-3e}$$

The whole external work:

$$L_{ext} = L_{ext}^{gable} + L_{ext}^{eccentr} + L_{ext}^{susp} + L_{ext}^{(N)} + L_{ext}^{(M_N)} \tag{3-4}$$

In the following the simplest solution is presented, we approximate the displacements with monomial functions:

$$u_D = U_1 \sin\frac{p}{l}z \tag{3-5a}$$

$$u_S = U_1 \frac{p^2}{l^2}\frac{D_y}{S}\sin\frac{p}{l}z \tag{3-5b}$$

$$\Phi = \Phi_1 \sin\frac{p}{l}z \tag{3-5c}$$

By substituting these functions into Eq (3-2) and (3-3a-e) and minimalizing by U_1 and Φ_1 the intensity of the critical load can be determined. This method is quite difficult. Another possible way of solution is when we determine the critical loads for every sub-cases ($q_{cr,i}$) and summarize them according to the following formula:

$$\frac{1}{q_{cr}} = \sum_{i=1}^{5} \frac{1}{q_{cr,i}} \tag{3-6}$$

3.1 Determination of the critical loads

In the following we will determine the critical loads for the above mentioned five cases.

Because of the internal work is the same in all five cases we will publish it in the first place.

By substituting Eq. (3-5a-c) into Eq. (3-2) we become the following formula:

$$L_{\text{int}} = \frac{\pi^2}{4l}\left(D_t + \frac{\pi^2}{l^2}D_\omega\right)\Phi_1^2 + \frac{\pi^4}{4l^3}D_y U_1^2\left(1 + \frac{\pi^2}{l^2}\frac{D_y}{S}\right) \tag{3-7}$$

3.11 The critical moment of the beam loaded in the shear center

By substituting the monomial functions (Eq. 3-5a-c) into the equation of the external work the following expression arises:

$$L_{ext}^{gable} = ql\left[\frac{3-p^2}{48}\Phi_1^2 b_1 + \frac{3+p^2}{24}\Phi_1 U_1\left(1+\frac{p^2}{l^2}\frac{D_y}{S}\right)\right] \tag{3-8}$$

From the equality of the internal and the external work the following partial derivatives can be calculated:

$$\frac{\partial}{\partial U_1}\left(L_{\text{int}} - L_{ext}\right) = 0 \tag{3-9}$$

$$\frac{\partial}{\partial \Phi_1}\left(L_{\text{int}} - L_{ext}\right) = 0 \tag{3-10}$$

From Eq. (3-9) and (3-10) the following system of linear equations arises:

$$\begin{bmatrix} -\frac{p^4}{2l^3}D_y\left(1+\frac{p^2}{l^2}\frac{D_y}{S}\right) & \frac{3+p^2}{24}ql\left(1+\frac{p^2}{l^2}\frac{D_y}{S}\right) \\ \frac{3+p^2}{24}ql\left(1+\frac{p^2}{l^2}\frac{D_y}{S}\right) & \frac{3-p^2}{24}b_1 ql - \frac{p^2}{2l}\left(D_t + \frac{p^2}{l^2}D_w\right) \end{bmatrix}\begin{bmatrix} U_1 \\ \Phi_1 \end{bmatrix} = 0 \tag{3-11}$$

Eq. (3-12) gives the non-trivial solution to determine the critical load:

$$\begin{vmatrix} -\dfrac{p^4}{2l^3}D_y\left(1+\dfrac{p^2}{l^2}\dfrac{D_y}{S}\right) & \dfrac{3+p^2}{24}ql\left(1+\dfrac{p^2}{l^2}\dfrac{D_y}{S}\right) \\ \dfrac{3+p^2}{24}ql\left(1+\dfrac{p^2}{l^2}\dfrac{D_y}{S}\right) & \dfrac{3-p^2}{24}b_1ql-\dfrac{p^2}{2l}\left(D_t+\dfrac{p^2}{l^2}D_w\right) \end{vmatrix}=0 \tag{3-12}$$

From Eq. (3-12) the following quadratic equation arises:

$$\frac{4}{p^4}\left(\frac{3+p^2}{24}\right)^2\frac{l^4}{\overline{D}_y}(q_{cr}^{gable})^2-\frac{3-p^2}{12}b_1q_{cr}^{gable}-\frac{p^2}{l^2}\left(D_t+\frac{p^2}{l^2}D_w\right)=0 \tag{3-13}$$

where

$$\overline{D}_y=\frac{1}{\dfrac{1}{D_y}+\dfrac{p^2}{l^2}\dfrac{1}{S}}$$

Eq. (3-13) is identical with the solution that can be found in the literature. But instead of the bending stiffness (D_y) we apply a "modified bending stiffness" that contains the effect of the shear deformation.

3.12 The critical bending moment of the beam loaded in point *P* and in the shear center with uniformly distributed forces

According to the steps used in section 3.11 the critical load can be calculated:

$$q_{cr}^{eccentr}=\frac{\pi^2}{tl^2}\left(D_t+\frac{\pi^2}{l^2}D_\omega\right) \tag{3-14}$$

3.13 The critical moment of the beam twisting around the suspension points

Using the steps described in the previous sections the critical load can be calculated:

$$q_{cr}^{susp}=\frac{\dfrac{\pi^4}{8}f}{\dfrac{l^4}{\pi^2\overline{D}_y}+\dfrac{t^2l^2}{D_t+\dfrac{\pi^2}{l^2}D_\omega}} \tag{3-15}$$

3.14 The critical load from the critical value of the axial force

Once again only the final solution is presented:

$$q_{cr}^{(N)}=\frac{2N_{cr}}{l\cot\alpha}. \tag{3-20}$$

3.15 The critical load from the eccentricity of the *N* axial force

The axial force (N) cause a constant moment ($M=aN$). The critical value of this moment (M_{cr}) can be calculated from Eq. (2-9). From the positive root of the equation the critical load can be calculated:

$$q_{cr}^{(M_N)} = \frac{2M_{cr}}{al \cot\alpha} \tag{3-21}$$

3.16 Summation of the critical loads

The critical load can be calculated by addition of the critical loads of the sub-cases according to the following formula:

$$\frac{1}{q_{cr}} = \sum_{i=1}^{5} \frac{1}{q_{cr,i}}$$

where $q_{cr,i}$ is the critical load derived from the sub-cases.

If one of the critical loads comes to a negative value (e.g. in case *b*) when the load acts under the shear center) then this load must be considered as infinite.

4. CONCLUSIONS

By solving the equations of the lateral torsional buckling of beams taking the shear deformation into account we arrived at the following simple result. The existing solutions in the literature (without shear deformation) can be readily used to determine the critical load with the following modification: the vertical bending stiffness of the beam must be modified according to the following formula:

$$\overline{D}_y = \frac{1}{\frac{1}{D_y} + \frac{p^2}{l^2}\frac{1}{S}}$$

where

S	is the shear stiffness of the beam in y direction,
l	is the distance between the suspension points of the beam.

5. REFERENCES

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