ANALYSIS AND OPTIMAL DESIGN OF BOUNDARY CONDITIONS

Vanda Pomezanski[1] *and Anna Vásárhelyi*[2]
Technical University of Budapest, Laboratory of Informatics
Mûegyetem rkp. 3., Budapest, H-1521

SUMMARY

The presented analysis is capable of determining the types, number and the places of the undefined boundaries. The applied theoretical analysis gives the possible variations of the conditions and an optimization method of determining the optimal location of that.

The applied finite element optimization method solves the minimal weight design of a structure and during the calculations the dimensions of the cross-section and the optimal supports.

Keywords: boundaries, calculus of variations, FEM, minimal weight design

1. INTRODUCTION

In a usual *structural analysis problem* the stresses and strains caused by external loads are calculated with continuous functions defined on the geometrical domain. The *structural design* is given, together with relevant properties of the material(s), cross-sections, shapes, layout, etc. to be used and the support conditions for the structure. For determining the structural response the relevant set of state equations, constitutive equations and compatibility conditions are used.

In the literature the design variables are taken into consideration as parameters which are changed than the corresponding changes of the structural response are determined via a repeated finite elements analysis (Eschenauer, 1997).

In this presentation the boundary conditions and the dimension of the cross-sectional area are the design variables, it means, the locations, numbers, and types of supports are unknown. The boundary variables are assumed to be discrete.

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[1]PhD student
[2]Professor in Structural Engineering

In the 2nd section the solution of the minimum problem of strain energy by a variational method is used to determine the necessary and sufficient boundary conditions.

In the 3rd section the result of that idea were used to design to minimal weight of the structure. Simple, discretized model of structures are presented as examples.

2. ANALYTICAL TASK OF BOUNDARY DETERMINATION

2.1 The minimum problem of strain energy

The optimization problem of beam:

$$p = \int_0^l \left[\frac{EI}{2} \left(u''(x) \right)^2 - P(x)u(x) \right] dx = \min, \qquad (1)$$

where E: Young modulus, I: moment of inertia, $P(x)$: load function, $u(x)$: displacement function and l: length of the beam.

The solution of the integral minimum problem by variational method is:

$$\int_0^l \underbrace{\left[EIu^{IV}(x) - P(x) \right]}_{\text{EULER's equation}} h(x)dx + \left[EIu''(x)h'(x) \right]_0^l - \left[EIu'''(x)h(x) \right]_0^l = Æ, \qquad (2)$$

residual (physical)
boundary conditions

essential (geometric)
boundary conditions

where h is an arbitrary function and $h \in C^1$ (Eschenauer, 1997).

2.2 The places of boundaries

If the boundary conditions of the calculated beam are given not at the end points, the structure must be divided into elements at the points where boundaries are possible. In this case in the Euler's differential equation the function of displacement *u(x)* depends on the location of the supports. It gives the possibility of having that location as an unknown variable. (Pomezanski, 1998.)

2.3 The types of boundaries

Fig.2.1 The model of the structure without supports

From the variational calculus the conditions for the boundaries are known:

$$EIu''(\ell)h'(\ell) = \varnothing\,, \qquad EIu'''(\ell)h(\ell) = \varnothing\,,$$
$$EIu''(\varnothing)h'(\varnothing) = \varnothing\,, \qquad EIu'''(\varnothing)h(\varnothing) = \varnothing\,. \tag{3}$$

The solution function, *u(x)* depends on the constants of integration so that can be written as $u(x,c_i)$ $i = 1,\ldots,4$. The function η*(x)* is arbitrary and satisfies the boundaries, so it can be chosen as $u(x,c_i)$. Finally there is only one function with 4 constants of integration as parameters:

$$EIu''(\ell,c_i)EIu'(\ell,c_i) = \varnothing\,, \qquad EIu'''(\ell,c_i)EIu(\ell,c_i) = \varnothing\,,$$
$$EIu''(\varnothing,c_i)EIu'(\varnothing,c_i) = \varnothing\,, \qquad EIu'''(\varnothing,c_i)EIu(\varnothing,c_i) = \varnothing\,. \tag{4}$$

The number of the possible solutions is ten:

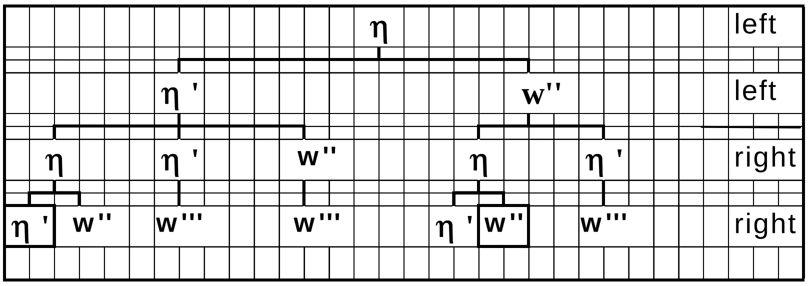

Fig.2.2 Support possibilities

The graph shows the 7 possibilities for the supports from left to right. The two signed are the symmetrical ones. Mirroring it from right to left there is other 7 possibility, where 4 are repetition (Pomezanski, 1998).

2.4 FEM

The analysis of a simple beam by finite element method in 2D gives the support vector and the rigidity matrix of sixth order. The minimum number of the supports is three. From the six chance there is

$\binom{6}{3} = 20$ kind of possible support solution.

In this calculation there is the two possibilities of supporting in the direction of x, what the equation of strain energy does not include. Dividing the previous result by two, the number of the possible supports is ten and the types are the same.

3. TASK OF BOUNDARY OPTIMIZATION

Structural optimization may be defined as the rational establishment of a structural design that is the best of all possible designs within a prescribed objective and a given set of geometrical and/or behavioural limitations. *Design variables* may be describe the configuration of a structure, element quantities like cross-sections, etc., and physical properties of the material. The possible design variables had been divided into six different classes. One if them contains the *supports* and loadings. In that discussion the idea is given up but results are not mentioned. (Eschenauer, 1997)

In this section the weight of the structure is minimized taking the cross-section area and the boundaries of the beam as unknown.

3.2 Problem formulation

The object of the research is a beam, supported at the two ends. In the center point there is a unit force. The types of the supports are unknown.

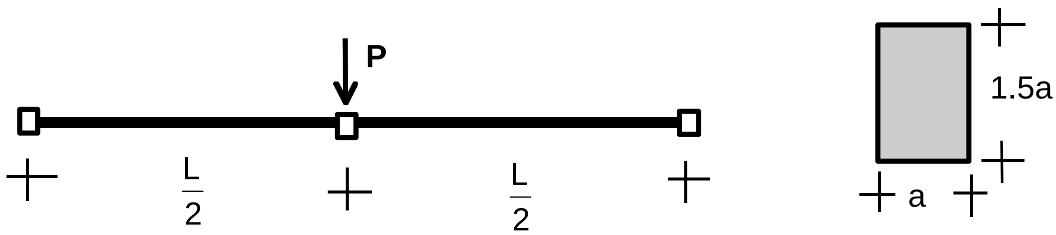

Fig. 3.1: *The model of the discretized structure and its cross-section*

The stiffness matrix of the structure contains the cross sectional area and inertia, $K(a^2, a^4)$.

The boundary conditions are defined in the vector ρ and its elements are added to the elements of the main diagonal of K.

$$K = K + E\rho, \tag{5}$$

where E is the unit matrix.

The state equation is:

$$[K(a^2, a^4) + E\rho]w = q \tag{6}$$

where w is the displacements vector and q is the external load vector.

In the optimization program the vector of the variables is X, where the first 9 elements ($X_{1,\dots,9}$) are the support variables (ρ), the second 9 elements ($X_{10,\dots,18}$) are the displacements (w) and the last one (X_{19}) is the width (a) of the cross-section.

3.3 The nonlinear optimization problem

The implementation of a sequential quadratic programming method for solving nonlinear optimization problems:

Minimize X_{19}

Subject to $\left(K\left(X_{19}, x_{19}^{3}\right) + EX_{1..9}\right)X_{10..18} - q = \varnothing$ equality constrains

$-10^{18} \le X_{1..9} \le 10^{19}$ boundary limits

$-0.05 \le X_{\substack{10,11\\13,14\\16,17}} \le 0.05$ displacement limits

$-0.10 \le X_{\substack{12\\15\\18}} \le 0.01$ rotation limits

$$0.05 \le X_{19} \le 0.50 \quad \text{cross-section limits} \qquad (7)$$

3.4 Example I. (Tab.3.1)

The starting structure is fixed at the joints 1 and 3. The hardness of the supports is the upper limit. The width of the cross-section is 0.31m. Solving the equation Kw=q we have the starting deflections. The displacement of the 2 joint in the direction y is greater than the limit one. The lower and upper limits of the supports variables of the 2nd joint is equal to zero.

No. joint		SUPPORT			DEFLECTION		
		x	y	z	x	y	z
1	start	0.1D+19	0.1D+19	0.1D+19	0.0D+00	0.5E-13	0.125E-12
	optimum	**0.1D+19**	**0.1D+19**	**0.1D+19**	0.0D+00	0.5D-13	0.125D-12
2	start	0.0D+00	0.0D+00	0.0D+00	0.0D+00	**0.8021E-01**	0.2164E-24
	optimum	**0.0D+00**	**0.0D+00**	**0.0D+00**	0.0D+00	0.5D-01	0.0D+00
3	start	0.1D+19	0.1D+19	0.1D+19	0.0D+00	0.5E-13	-0.125E-12
	optimum	**0.1D+19**	**0.1D+19**	**0.1D+19**	0.0D+00	0.5D-13	-0.125D-12

Tab. 3.1 Starting and optimal values of example I.

The solution: None of the supports has been changed. The displacement of the middle joint went to the limit value. The optimal width of the cross-sectional area is 0.3489m.

3.5 Example II. (Tab.3.2)

The starting structure is fixed at the joint 1 and the rotation of the joint 3 is tied. The hardness of the supports is the upper limit. The width of the cross-section is 0.31m. More of the calculated displacements are out of limits. The lower and upper limit of the support variables of the 2nd joint is equal to zero.

No. joint		SUPPORT			DEFLECTION		
		x	y	z	x	y	z
1	start	0.1D+19	0.1D+19	0.1D+19	0.0D+00	0.10E-12	0.42E-13
	optimum	**0.1D+19**	**0.1D+19**	**0.1D+19**	**0.63D-16**	**0.52D-13**	**0.66D-13**
2	start	0.0D+00	0.0D+00	0.0D+00	0.0D+00	-0.24E+00	-0.16E+00
	optimum	**0.0D+00**	**0.0D+00**	**0.0D+00**	**0.24D-14**	**0.50D-01**	**-0.81D-01**
3	start	0.0D+00	0.0D+00	0.1D+19	0.0D+00	-0.64E+00	-0.21E-12
	optimum	**0.71D+04**	**-0.11D+08**	**0.1D+19**	**0.328D-14**	**-0.37D-02**	**-0.15D-12**

Tab. 3.2 Starting and optimal values of example II.

The solution: Some of the supports has been changed. The fixed ones at the upper limit and the unchangeable middle ones has not. The changeable unfixed ones have got an elastic support. The displacement of the joints went to or under the limit values. The optimal width of the cross-sectional area is 0.2262m.

3.6 Example III. (Tab.3.3)

The starting structure is fixed at the joint 1 and the rotation of the joint 3 is hindered. The value of the starting support variable is less, 10^{13}. The width of the cross-section is 0.31m. More of the calculated displacements are out of limits. The lower and upper limit of the support variables of the 2nd joint is equal to the other ones, as written in (7).

No. joint		SUPPORT			DEFLECTION		
		x	y	z	x	y	z
1	start	0.1D+13	0.1D+13	0.1D+13	0.0D+00	0.10E-06	0.42E-07
	optimum	**0.99D+12**	**0.99D+12**	**0.99D+12**	**-0.16D-15**	**-0.19D-07**	**-0.86D-07**
2	start	0.0D+00	0.0D+00	0.0D+00	0.0D+00	-0.24E+00	-0.16E+00
	optimum	**-0.72D+07**	**-0.20D+07**	**-0.76D+06**	**-0.99D-12**	**-0.50D-01**	**-0.10D+00**
3	start	0.0D+00	0.0D+00	0.1D+13	0.0D+00	-0.64E+00	-0.21E-06
	optimum	**0.64D+07**	**0.44D+06**	**0.99D+12**	**0.36D-12**	**-0.50D-01**	**-0.26D-06**

Tab. 3.3 Starting and optimal values of example III.

The solution: All of the supports has been changed. The starting ones became to be a little softer. The other ones got much elasticer supports. In the middle joint the supports have the opposite direction. The displacement of the joints went to or under the limit values. The optimal width of the cross-sectional area is 0.1895m. The result structure is not symmetrical.

4. CONCLUSION

The presented analysis is capable of determining the types, number and the places of the undefined boundaries. The applied theoretical analysis gives the possible variations of the conditions and an optimization method of determining the optimal location of that.

The applied finite element optimization method solves the minimal weight design of a structure and during the calculations the dimensions of the cross-section and the optimal supports. The results of the optimization shows that the different starting values of design variables presents different solutions. It happens because the cross-sectional area and inertia cause the nonconvexity of the feasible set and the computer algorithm find different local extremal values.

The presented method and results show that the structural optimization with boundary optimization is a solvable problem, but the global optimum can not be guarantied.

5. REFERENCES

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