FOLDABLE BAR STRUCTURES ON A SPHERE

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SUMMARY

Some bar structures have already been invented that have a closed circular shape and are foldable along their perimeter. These mechanisms usually belong to one of the two following basic categories: either they contain exclusively hinges and pivots (scissor-like hinges) but can move only along a plane, or they are able to move along other surfaces (mainly a sphere) but have some sliding mechanisms or many degrees of freedom. Our main objective was to find a structure that has the advantages of both categories above mentioned. In the followings some types of spherical bar structures are presented which have no sliding mechanisms but small number of independent motions.

Keywords: pivot, finite motion, infinitesimal motion, Jacobian matrix, spherical mechanism

1. INTRODUCTION

Consider two sets of parallel straight rods connected by pivots (Fig. 1a) - it is a system of a single finite motion. This character will be preserved if the system is multiplied and connected to each other (Fig. 1b) or even if it is made of angulated rods (Fig. 1c), (You and Pellegrino, 1997). The last solution makes possible the construction of a closed ring that is able to change its perimeter.

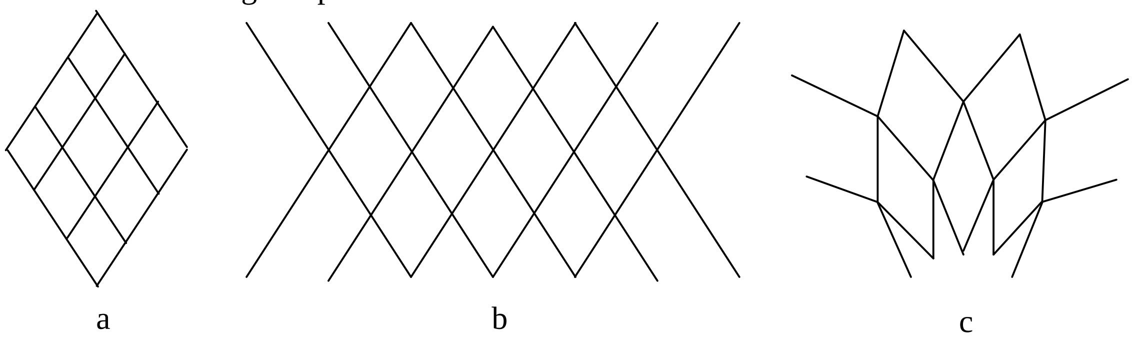

Fig. 1: Basic planar mechanisms

A serious problem is, however, that the same topology produces no mechanism on a sphere: while in the plane two bars connected by many others can form a finite

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mechanism, it seems to be impossible that two rigid figures on a sphere, connected by more than two great circle arcs could have finite internal motions (Fig. 2). Note that in case of central symmetry two antipodal connections are not considered different ones.

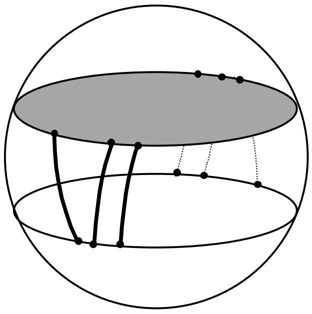

Fig. 2: Spherical structure with 3 connecting arcs (not finite mechanism)

For this reason, there must be found other topological solutions in the spherical case. Since in the followings will take place some considerations about spherical geometry, it looks useful to introduce some simplifications. The rotational symmetry of our new structure should be preserved, so its axis marks a special direction like that passes through the two poles of the globe - consequently there will be written 'meridian' and 'parallel', 'latitude' or 'longitude' if necessary.

There are now two basic problems to solve. First of them is, how to ensure this rotational symmetry in each position of the structure? Since the existence of a similar 'automatic' solution to that of the planar case is ignored so far on a sphere, the most evident choice is to build the whole mechanism upon a fixed base circle, a parallel with equidistant pivots.

The other question is about the spherical character of the mechanism: what can ensure the hinges or pivots to remain along the same sphere? The solution is that all the connecting elements (referred as 'bars') are great circle arcs and each rotational axis of the hinges and pivots passes through the centre of the sphere (Fig. 3).

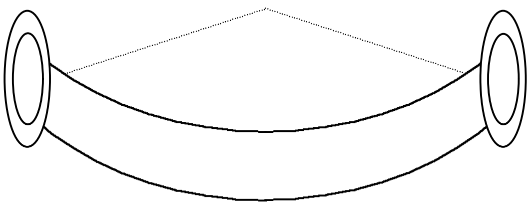

Fig. 3: Spherical connecting element ('curved bar')

2. THE STRUCTURE

If n fixed hinges (P_{10}, P_{20},... P_{n0}) are considered to be at the same latitude, in such a way that they form a system of n-order rotational symmetry, it means practically a base circle with n pivots. Connecting to each pivot a great circle arc of equal length (referred as 'arm' in the followings), the structure called 'open' if the arms' planes are perpendicular to that of the base circle, 'closed' if they coincide. The next step is to ensure the same inclination of these arms to the base circle. For this purpose consider a new pivot on each arm (P_{11}, P_{21},... P_{n1}) in such a way that the distances $P_{i0}P_{i1}$ (i=1, 2,... n) be equal, a closed and rigid circle (referred as 'ring' in the followings) with n equidistant pivots (denoted as Q_{11}, Q_{21},... Q_{n1}), finally n connecting bars of equal length between each pair $P_{i1}Q_{i1}$ (i=1, 2,... n). In this case, if the plane of the ring is parallel to that of the base circle, the rotational symmetry of arms is guaranteed (Fig. 4).

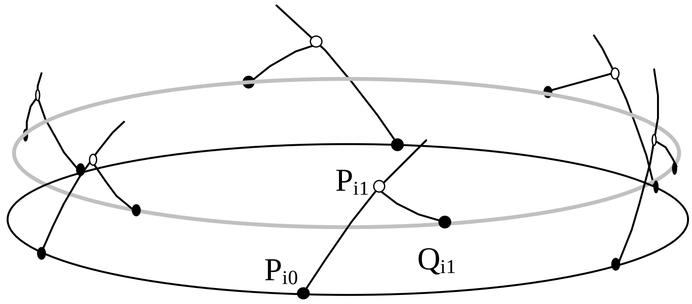

Fig. 4: Arms, rings and connecting bars on a sphere

Here comes up a new problem because this horizontal position of the ring is not ensured. It is easy to see that - within a certain domain of the surface - all positions of the ring are possible, since it may be connected to any fixed points by a set of two-bar connections (Fig. 5a). Our solution is based on the conjecture that these 'non-symmetrical' cases produce a fairly special distribution of arms' inclination angles which cannot coincide in case of different $P_{i0}P_{i1}$ or $P_{i1}Q_{i1}$ lengths. Consequently, the proposed solution is the following: consider another set of pivots on the arms (P_{12}, P_{22},... P_{n2}), another ring with n equidistant pivots (Q_{12}, Q_{22},... Q_{n2}) and another length of connecting bars. Our statement is that now there is a unique geometrically possible configuration: where the rotational symmetry exists (Fig. 5b).

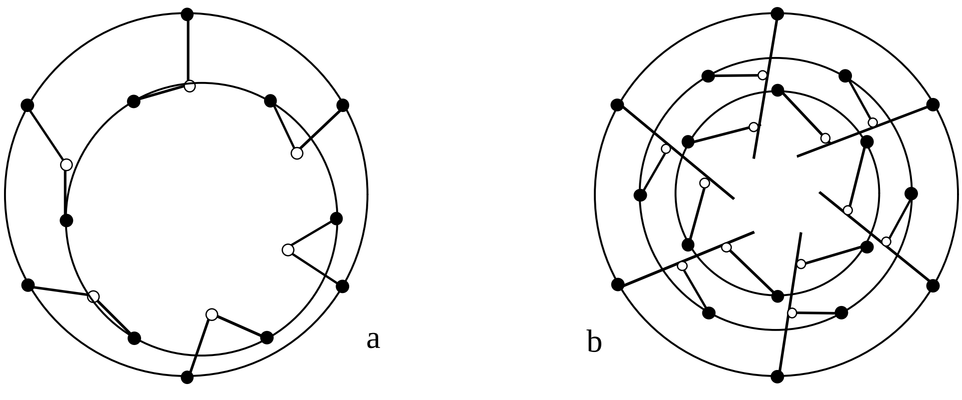

Fig. 5: Geometrically possible configurations at one (a) and more (b) rings

It is still a question, however, whether two rings are sufficient or not. In order to make a decision, let be supposed that the structure has n arms and r rings. Consider first only the n arms (connected to the base circle) and n×r connecting bars - this compound means $f=n+nr=(r+1)n$ degrees of freedom. If two neighbouring arms are linked by a piece of the corresponding ring, it is necessary to have n-1 pieces to link all the n arms to each other. To ensure a connection between two neighbouring ring pieces that does not allow for rotation means another n-2 constraints (the ring obtained in such manner is open but the last piece with the two ‘stiffened’ hinges mean just three more degrees of kinematical overdeterminacy, without impeding any other motion). Multiplied by the number of rings, the formula is $c=r(2n-3)$ for the number of constraints. If $f-1\leq c$ is satisfied, it is possible for the structure to have only one finite motion. From the solution of the following equation

$$n(r+1)-1\leq r(2n-3) \tag{1}$$

is obtained that r must be at least 2 and it implies 5 as a minimum for n, if r=3 then n must be not less than 4, finally, if $n\leq 3$, the equation has no solution again. For the following analysis the r=2, n=5 case was chosen.

3. NUMERICAL METHOD

The following control method is based on the analysis of the Jacobian matrix of the whole structure (Kovács, Hegedûs and Tarnai, 1997). Since all the bars are supposed to be rigid, their length must be constant (in this model each curved bar is represented by a straight rod). This condition implies that all F_{ij} constraint functions - which express the elongation of the corresponding bars - must have a zero value:

$$F_{ij}=\sqrt{(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2}-l_{ij}=0\,, \tag{2}$$

where x_i, y_i, z_i and x_j, y_j, z_j are the coordinates of both end-points of the straight bar whose length is l_{ij}.

All changes in bar lengths can be approximated by their Taylor-series. Considering a first-degree approximation, e.g. for the ‘

$$F_{ij}+\Delta F_{ij}\approx F_{ij}+\sum_k\frac{\partial F_{ij}}{\partial X_k}\Delta v_k\,, \tag{3}$$

where X_k represents all the coordinates in a given position and Δv_k all the corresponding infinitesimal displacements.

The second member of (3) can be written as **Jv** where **J** is the Jacobian matrix of the structure. Since ΔF must be zero for each bar, the following equation must be satisfied:

$$\mathbf{Jv}=\mathbf{0}. \tag{4}$$

Apart from the trivial solution (**v**=**0**) it is possible only when the rank of **J** is less than the number of its columns.

There is a possibility to reduce the number of columns in **J**. It has been already mentioned that all the hinges and pivots move along a sphere, so the x, y, z Cartesian coordinates can be substituted by only two spherical coordinates: φ for the longitude and θ whose value is equal to π/2-latitude. After this substitution for any point P:

$$P_i(x_i,y_i,z_i)=P_i(\sin\theta_i\cos\varphi_i,\sin\theta_i\sin\varphi_i,\cos\theta_i)=P_i(\varphi_i,\theta_i)\,, \tag{5}$$

from substitution in (2) and derivation two new formulae are obtained:

$$\frac{\partial F_{ij}}{\partial\varphi_i}=\frac{1}{l_{ij}}\sin\theta_i\sin\theta_j\sin(\varphi_i-\varphi_j)\,, \tag{6a}$$

$$\frac{\partial F_{ij}}{\partial\theta_i}=\frac{1}{l_{ij}}\left(\sin\theta_i\cos\theta_j-\cos\theta_i\sin\theta_j\cos(\varphi_i-\varphi_j)\right). \tag{6b}$$

In order that not only the existence but other characteristics of an infinitesimal mechanism be verified, the singular value decomposition of **J** will be made. It means that if the structure contains b bars and h hinges (pivots), **J** is a b×2h matrix such that

$$\mathbf{J}_{(b\times 2h)} = \mathbf{U}_{(b\times b)}\,\mathbf{S}_{(b\times 2h)}\,\mathbf{V}^{T}_{(2h\times 2h)}. \tag{7}$$

Here **S** is a matrix that contains all its non-zero elements - the singular values of **J** - in its main diagonal, **U** and **V** are orthogonal matrices, that is, $\mathbf{V}^T\mathbf{V}=\mathbf{I}$ and all columns and rows of **V** are linearly independent. It means that if **J** is multiplied from right by the k-th column of **V** (let be denoted $\mathbf{v}^k$), the product is equal to $s_k\mathbf{u}^k$, where s_k is the k-th singular value and $\mathbf{u}^k$ is again the k-th column of **U**. Consequently, this product $\mathbf{Jv}^k$ can only be zero if either s_k is zero or k>min(b, 2h) and in these cases each corresponding $\mathbf{v}^k$ represents an independent system of infinitesimal displacements.

The singular value decomposition of **J** provides a method for decide whether a system of displacements is finite or not. Let $\mathbf{X}_{(i)}$ be a vector that contains all coordinates of the structure used during the i-th calculation and $\mathbf{v}_{(i)}$ the vector of the corresponding displacements. Since **v** can freely be multiplied by any non-zero constant, it is useful to choose for this purpose a 'relatively small' number (if the values of **v** are big, the approximation will be very rough but in opposite case the multiplied inaccuracies of calculations will cause false results). Adding this new $\mathbf{v}_{(i)}$ to the actual $\mathbf{X}_{(i)}$, an $\mathbf{X}_{(i+1)}$ is obtained, so it is possible to compile a new Jacobian matrix and calculate $\mathbf{v}_{(i+1)}$, etc. If the analysed motion is finite, the s_k values that were zero at the beginning, will preserve this value (apart from calculation inaccuracies), while in case of infinitesimal motions these singular values becomes different from zero in a few steps.

Note that there is a possibility for a conversion from Cartesian coordinates to spherical ones if needed (principally as input data of any graphic display). For example:

$$dX_i(\theta_i,\varphi_i) \approx \frac{\partial X_i(\theta_i,\varphi_i)}{\partial \theta_i}d\theta_i + \frac{\partial X_i(\theta_i,\varphi_i)}{\partial \varphi_i}d\varphi_i\,, \tag{8}$$

substituting the formulae of (5)

$$\Delta x_i = \cos\theta_i \cos\varphi_i \Delta\theta_i - \sin\theta_i \sin\varphi_i \Delta\varphi_i\,, \tag{9a}$$

$$\Delta y_i = \cos\theta_i \sin\varphi_i \Delta\theta_i + \sin\theta_i \cos\varphi_i \Delta\varphi_i\,, \tag{9b}$$

$$\Delta z_i = -\sin\theta_i \Delta\theta_i \tag{9c}$$

are obtained.

The last topic to be discussed here is how to model a pivot with simple pin-ended bars. In the present analysis additional bracing was applied that implied introduction of new nodal points (when a pivot is an internal point of a great circle arc). Bracings of arms and rings are shown in Fig. 6.

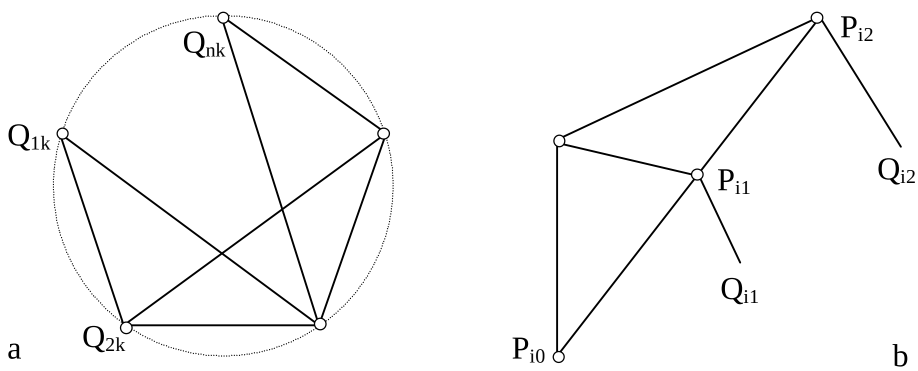

Fig. 6: Bracing of arms (a) and rings (b)

4. RESULTS

In order to the execution of these calculations, a computer program was written in C language that is able to analyse the Jacobian matrix of a foldable bar structure in any position between the open and closed one. Its main input data are the coordinates and topology of the network and the step of motion increment. Its output data are the **v** vectors that belong to independent finite and infinitesimal mechanisms - in any positions.

Considering the bracings, our structure contained 49 bars and 25 movable nodal points, consequently its Jacobian matrix was of 49×50. This means that one motion was evident but it can also be seen from view (the pure rotation of the rings that implies the elevation of each arm).
The executed analysis proved that this system has two other - infinitesimal - motions. It leads to two basic consequences:
(i) in the neighbourhood of the symmetrical position there are no other geometrically possible configurations, but
(ii) the structure - due to these infinitesimal motions - probably does not have the sufficient rigidity for practical application without some special support.

5. OTHER POSSIBILITIES

Rigid circles - rings - were necessary in the previous model in order that rotational symmetry be preserved. It produced an interesting structure and gave opportunity to introduce a method of kinematical analysis of spherical bar structures. From point of view of the practice, however, exists another solution that has less doubt of applicability (Fig. 7).

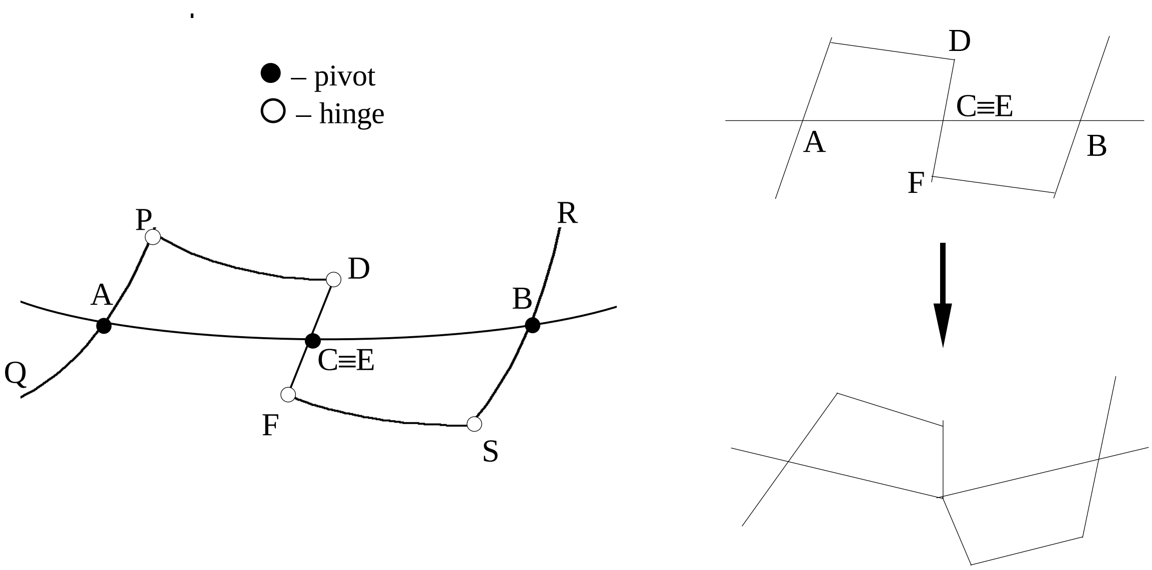

Fig. 7: Alternative mechanism

If we consider two great circle arcs of equal length (PQ and RS) with a fixed pivot in their mid-point (A and B), we can guarantee the identical inclination of both elements to the AB meridian, where C is an internal point such that AC=CB. Let DEF be another segment of arbitrary length where DE=EF. If we connect ACB and DEF by a pivot at their mid-points (C≡E) and we put two simple curved bars of equal length between PD and FS, the mechanism is able to "copy" the angle CAP to CBS with given length AB.

This structure, however, has a restriction that the base circle must be also a great circle. To avoid this restriction, let's consider a base circle of arbitrary radius and let's fit tangent great circle arcs to the circle at each pivot. If the intersection point between A and B is C, now we must connect to C by another pivot a broken line (DEF), where ∠ACB=∠DEF. This configuration will also preserve the local central symmetry where ∠CAP and ∠CBS are equal - consequently, these mechanisms will preserve their rotational symmetry as well.

6. CONCLUSIONS

There was presented a computational model for analysing the internal motions of any spherical bar structures. It has been proved the existence of some radially deployable spherical mechanisms that contain no sliding connections and are able to be folded along their perimeter with only one independent finite motion. With the help of some results of our analysis it was possible to draw some conclusions about the behaviour of the presented structures and has also been shown that there is a real perspective of their application in the practice.

Further investigations are necessary to decide what type of kinematical overdeterminacy can be reached on a spherical mechanism in order to minimize the displacements under loading; this may require a detailed analysis on the basic differences between spherical and planar mechanisms.

7. ACKNOWLEDGEMENTS

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8. REFERENCES

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