TORSION OF REINFORCED CONCRETE BEAMS

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SUMMARY

In the first part of the following paper we will introduce the most important methods for analyzing a reinforced concrete member. In the second part we will present the results of the experiments made on 15 reinforced concrete and unreinforced concrete beams loaded by pure torsion, and compare the results obtained by the above mentioned methods with the results calculated by Eurocode2 and the Hungarian Code.

Keywords: reinforced concrete, torsion, cracks, rotation of cracks

1. INTRODUCTION

In structural analysis the effects of torsion are usually neglected and only bending, shear, and axial forces are taken into account. Why should we analyze torsion after all? There are numerous reasons. First of all in some structures (e.g. L shaped beams) its effects should be taken into consideration. Secondly even small torsional moments can raise considerable stresses and so can change the response of the whole structure. Thirdly with greater computational capacity (3D analysis) torsional load can easily be calculated. Moreover the combination of torsion with other loads is still not clear.

Analyzing the torsional behavior of a reinforced concrete member we can be divided into three main stages. The first is the pre-cracking stage where it is possible to assume a homogenous concrete cross section. The second is the post cracking stage where a method based on Rausch's space truss analogy is normally used. The third stage is the stage where the crack occurs. This stage has more importance in the case of complex loading when it is necessary determine an interaction surface.

2. PRE-CRACKING BEHAVIOR

To calculate the pre-cracking behavior and the cracking load there are three basic methods. The elastic theory of torsion was presented by Saint-Venant in the middle of the last century. It claims that for the crack load:

$$T_e = \alpha \cdot x^2 \cdot y \cdot f_t' \qquad \alpha = 0.208$$

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The problem with this equation is that the obtained results are unconservative by approximately 50%. Nylander first used the plastic method in 1955 to correct this huge difference. He used a similar equation but he utilized Nadai's plastic coefficient into is instead of Saint-Venant's elastic one:

$$T_p = \alpha_p \cdot x^2 \cdot y \cdot f_t' \qquad \alpha_p = \left(\frac{1}{2} - \frac{1}{6} \cdot \frac{x}{y} \right)$$

There are two problems with this equation. The first problem is obvious: the failure is definitely not plastic. The second is that it does not account for the size effect.

The third method is the skew-bending theory. This is based on observations of torsion tests. Here the first crack appears on the front face and inclines at 45% to the axis of the beam. It gradually widens and progresses across the top of the beam until, ultimately, the concrete crushes on the back face. This failure is similar to that of a unreinforced concrete flexural beam and, therefore, reveals a bending-type failure. The ultimate strength equation is as follows:

$$T_s = \frac{x^2 \cdot y}{3} \cdot f_r$$

Comparing the elastic theory, the plastic theory and the skew-bending theory it can be seen that they all contain $x^2 \cdot y$. The only differences lie only in the non-dimensional coefficients and material constants.

3. POST-CRACKING BEHAVIOR

In recent years many theories have been developed for calculating the torsional strengths of members with both longitudinal steel and stirrups in the post-cracking range. These theories can be roughly divided into two types: the space-truss analogy and the skew-bending theory. The former is the oldest and more commonly used in the different codes, the fundamentals of this theory are described below first.

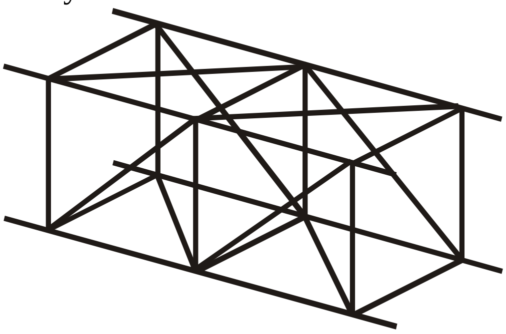

Fig. 1. The scheme of the space truss assumed in Rausch's theory

3.1. Space-truss analogy

The first theory of reinforced concrete subjected to torsion was proposed by Rausch in 1929 in the form of a Ph.D. thesis. The first simplification is that the examined cross section has an arbitrary shape and is assumed to be hollow. This assumption is based on theoretical foundations: the stresses in the core of the cross section are much lower than at the perimeter. Later experiments also proved this assumption to be true. They showed that the ultimate torque of hollow and solid members are more or less the same. After cracking, the concrete is separated by cracks into a series of helical members. These helical concrete members are assumed to interact with the longitudinal steel bars and the hoop steel bars to form a space truss.

We shall now analyze the forces in the members of this space truss. First, the internal forces in the longitudinal bars, the hoop bars, and the diagonal concrete struts are denoted X, Y, and D, respectively. The force at each joint representing the shear flow is designated F. Second, each force is labeled in sequence with subscripts from 1 to n along the periphery of the cross section. Third, we notice that the internal forces X, Y, and D must be identically distributed in each cell in the longitudinal direction.

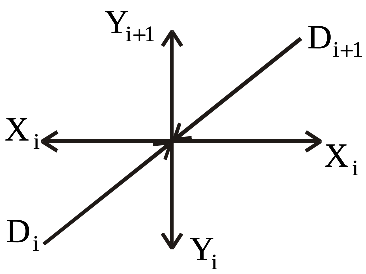

Fig.2. The forces acting in the joint C_{r+1}

If we take the equilibrium of joint C_{r+1} in the longitudinal direction we can see that the forces in the concrete struts are equal (D = constant). Secondly consider the equilibrium of the joint in the lateral (or tangential) direction. Now we can take the same observation for the forces in the stirrups (Y = constant). By studying the equilibrium of joint C_{r+1} in the radial direction, which is perpendicular to both the longitudinal and lateral directions, we can be see that:

$$D = X \cdot \cos\alpha + Y \cdot \sin\alpha$$

Now if we take a look at the sum of the forces outside a joint(D and X) in the tangential direction we can write:

$$F = D \cdot \sin\alpha = (X \cdot \cos\alpha + Y \cdot \sin\alpha) \cdot \sin\alpha$$

Finally we can study the equilibrium of the whole cross section. The internal torques contributed by F_1 to F_n must be equal to the external torque T. Denoting the distance from the force F_r to the axis of twist as a_r, the internal torque due to F_r is $F_r \cdot a_r$. Summing the internal torque for F_1 to F_r and noticing that all the F forces have constant value:

$$T = \sum_{r=1}^{n} F_r \cdot a_r = F \cdot \sum_{r=1}^{n} a_r = (X_r \cdot \cos\alpha + Y_r \cdot \sin\alpha) \cdot \sin\alpha \cdot \sum_{r=1}^{n} a_r$$

If we denote the spacing of longitudinal bars as *b* and the enclosed area inside a stirrup as A we can write:

$$T = (X_r \cdot \cos\alpha + Y_r \cdot \sin\alpha) \cdot \sin\alpha \cdot \sum_{r=1}^{n} \frac{2 \cdot A_r}{b} = (X_r \cdot \cos\alpha + Y_r \cdot \sin\alpha) \cdot \sin\alpha \cdot \frac{2 \cdot A}{b}$$

Assuming that both the longitudinal bars and the stirrups are yielding at the ultimate torque and that $\alpha = 45°$ we can obtain two values for the ultimate torsional moment:

$$T_{U1} = \frac{2 \cdot A \cdot A_l \cdot f_{ly}}{s} \qquad T_{U2} = \frac{2 \cdot A \cdot A_t \cdot f_{ty}}{s}$$

where: A_l is the cross-sectional area of one longitudinal bar
f_{ly} is the yield strength of longitudinal bars
A_t is the cross-sectional area of one stirrup
f_{ty} is the yield strength of stirrups

3.2. Modifications on Rausch's Theory

In 1935 Andersen pointed out that Rausch's truss analogy assumed uniform stress along all the reinforcement in a member subjected to torsion. This assumption of uniform stress contradicts Saint-Venant's stress distribution for all types of cross sections except circular. In the case of a rectangular section Saint-Venant's stress distribution requires that maximum stress occur at the wider face and decrease zero at the corner. In view of this non-uniform stress distribution in the steel the torsional resistance of reinforcement should be less effective than that predicted by Rausch's equations. Consequently, Andersen suggested that Rausch's results should be modified by an efficiency coefficient that is less than unity. Later Cowan obtained a slightly different equation. His result was based strictly on Saint-Venant's stress and strain distribution for rectangular cross sections. His final equation was:

$$T_{UC} = T_e + 1.6 \cdot \frac{x_1 \cdot y_1 \cdot A_t \cdot f_{ty}}{s}$$

where: T_e is the torsional resistance of unreinforced concrete taken as the elastic torque
x_1 and y_1 are the length of the arms of the stirrups

Cowan's method, however, contains an error since it is based on Saint-
The validity of Saint-Venant's theory is doubtful for the post-cracking range. Later tests have shown that the stress along the reinforcement are essentially uniform and do not vary according to Saint-Venant's distribution. Even so, the concept of using an efficiency coefficient to improve Rausch's torsional resistance has been widely accepted.

3.3. The skew-bending theory

This theory is much younger than Rausch's. It was first proposed by Lessig in 1958. The basic characteristic of the skew-bending theory is the assumption of a skew failure surface. The surface is initiated by a helical crack on three faces of a rectangular beam,

while the ends of this helical crack are connected by a compression zone near the fourth face.
The internal torsional resistance arises from three sources: the axial forces of the stirrups, the shear-compression force of concrete, and the dowel forces of the longitudinal bars. After some algebra Hsu obtained the following equation in 1968:

$$T_{US} = \frac{x^2 \cdot y}{3} \cdot \left(2.4 \cdot \sqrt{f_c'}\right) + \sqrt{m} \cdot \frac{f_{ly}}{f_{ty}} \left(1 + 0.2 \cdot \frac{x_1}{y_1}\right) \cdot \frac{x_1 \cdot y_1 \cdot A_t \cdot f_{ty}}{s} = T_c + \alpha_t \cdot \frac{x_1 \cdot y_1 \cdot A_t \cdot f_{ty}}{s}$$

where: $m = \dfrac{2 \cdot \underline{A_l} \cdot s}{2 \cdot A_t \cdot (x_1 + y_1)}$

x and y are the outer dimensions of the cross section
A_t is the cross-sectional area of one stirrup
$\underline{A_l}$ is the area of all the longitudinal bars
f_{ty} is the yield strength of stirrups
f_{ly} is the yield strength of longitudinal bars
f_c' is the specified compressive strength of concrete
s is the stirrup spacing

Comparing the final equations obtained by the two different theories we can observe that although they are not identical their structure is similar.

3.4. Codes

Both the Eurocode2 and the Hungarian Code are based on Rausch's space truss theory and the design method is the same in both. The engineer has calculate three different ultimate values, for the longitudinal bars, for the stirrups, and for the diagonal concrete struts. There are, however, two main differences. The first difference is the calculation of the ultimate torsional moment in connection with the concrete. The second and more important difference is that while the Hungarian Code uses a constant value (α = 45°) for the angle of cracks and hence the concrete diagonals, the Eurocode2 allows it to change (0.4 = ctgα = 2.5). This angle appears in all three prediction equations proposed by the Eurocode2 and is hidden in the equations of the Hungarian Code as well.

	Eurocode2	Hungarian Code
Longitudinal bars	$T_{Rd3} = 2 \cdot A_k \dfrac{A_{sl} \cdot f_{yld}}{u_k} \cdot tg\alpha$	$T_{UH1} = 2 \cdot A \cdot \dfrac{A_{st} \cdot \sigma_{sl}}{s}$
Stirrups	$T_{Rd2} = 2 \cdot A_k \dfrac{A_{sw} \cdot f_{ywd}}{s} \cdot ctg\alpha$	$T_{UH2} = 2 \cdot A \cdot \dfrac{A_{st} \cdot \sigma_{st}}{s}$
Concrete	$T_{Rd1} = \nu \cdot f_{cd} \cdot t \cdot 2 \cdot A_k \cdot \dfrac{ctg\alpha}{1 + ctg^2\alpha}$ $\nu = 0.7 \cdot (0.7 - \dfrac{f_{ck}}{200})$	$T_{UH3} = 0.3 \cdot W_t \cdot \sigma_c$ $W_t = \dfrac{b^2 \cdot a}{3 + 1.8 \cdot b/a}$

Table 1. Ultimate torsional moments according to the Codes

4. EXPERIMENTS

At the Technical University of Budapest a series of tests were carried out with plain concrete and reinforced concrete beam specimens loaded with pure torsion. The supports allowed both rotating and lengthening of the elements. All the beams had the same outer dimensions: 130·130·2000. There were five sets of beams each consisting of three specimens. The reinforcement ratios were varied both in the longitudinal and the transversal directions. The difference in the reinforcements can be seen in Table 2. To apply the load, steel arms were attached to both ends of the beams (See Fig. 3.). The actual loading was done by a hydraulic jack at one side but the force was measured on both sides. The twist of the middle section (700 mm) was also measured by the shift of attached arms. The force and the difference in the deflections were recorded by a plotter to obtain a torque-twist diagram.

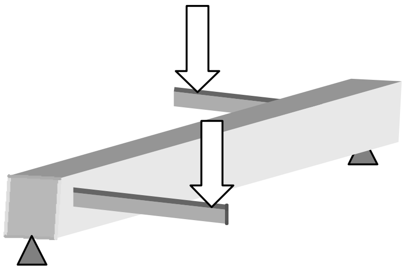

Fig.3. The layout of the torsional tests

Three methods were used to detect the cracking torsional moment. The first method was the visual observation of the surface at every load step. As the cracks began at the middle of the beams they were easily observable. The second method was the observation of the torque-twist diagram. At cracking the torsional rigidity decreased significantly and therefore the slope of the diagram decreased. The third method was the observation of the sound of the cracking. The ultimate torsional moment was taken as the maximum moment reached throughout the test. At every load step the cracks were marked. After the tests these marks were drawn on paper so that the angle between the cracks and the axis of the beam could be measured.

Beam	Stirrups	Longitudinal bars
A	*unreinforced*	
B	ϕ6/130	4ϕ6
C	ϕ6/65	4ϕ6
D	ϕ6/130	8ϕ6
E	ϕ6/65	8ϕ6

Table 2. Reinforcement details of the tested beams

Stirrups were closer together at the beam ends to ensure failure in the middle section. The concrete used was C20 in all cases. The tensional strength was obtained by: $f_{ct}=0.3 \cdot f_c^{2/3}$ The steel was B.38.24 ($f_y = 240$ N/mm^2).

5. RESULTS

5.1. Cracking moment

The values of the measured and calculated cracking moment are given in Table 3and plotted in Fig. 1.The bald values in Table 3. Indicate that the calculated values that are smaller than the measured ones. The diagrams illustrate the discrepancy between the calculated and the measured values. The points that are below the solid lines represent underestimation. The points above the diagonal lines represent an overestimation of the loads. It is obvious from Fig. 4. that the elastic method, almost in all cases, gives a conservative result and that the plastic method always gives an unconservative result. It is also apparent from Fig. 4. that the best prediction is given by the skew-bending theory. So for design purposes, the elastic method seems to be the most conservative and for modeling, where we must be close to the exact value, the best is the skew-bending theory.

	Measured	Elastic	Plastic	Skew-bending
A-1	1,61	**1,13**	1,81	**1,54**
A-2	1,69	**1,32**	2,11	1,79
A-3	-	1,24	1,98	1,68
B-1	-	1,45	2,32	1,97
B-2	1,40	1,51	2,42	2,05
B-3	1,64	**1,43**	2,30	1,95
C-1	1,79	**1,47**	2,36	2,01
C-2	1,89	**1,47**	2,36	2,01
C-3	1,63	**1,41**	2,26	1,92
D-1	1,73	**1,29**	2,07	1,76
D-2	1,69	**1,49**	2,40	2,04
D-3	1,69	**1,47**	2,36	2,01
E-1	2,25	**1,55**	2,49	2,12
E-2	1,69	**1,48**	2,38	2,02
E-3	2,06	**1,52**	2,44	2,07

Table 3. The measured and the calculated cracking moments (kNm)

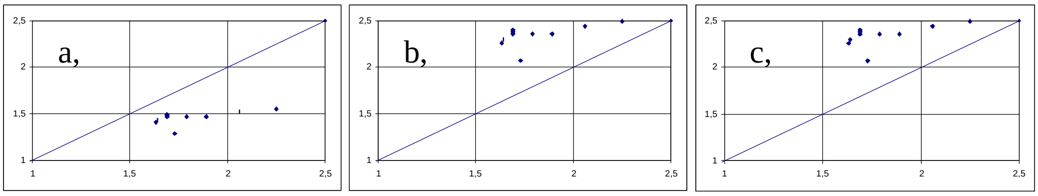

Figure 4. Measured versus calculated cracking moments (a-elastic theory, b-plastic theory, c-skew-bending theory) (kNm)

5.2 Ultimate moment

The measured and calculated ultimate moments are shown in Table 4. The results for the unreinforced beams (A-1, A-2, A-3) can not be found there because for them cracking means automatic failure. Because the beams in the same set had the same parameters the average of the three is also listed. The next two columns contain the values for the ones calculated by Rausch's theory assuming 45° cracks. These are the same as in the Hungarian Code and in the Eurocode2. The first of the columns refers to the yielding of the stirrups and the second is to the yielding of the longitudinal bars. In the last three columns the values are different for every beam because the tensile strength of the concrete appears in the equations which were used to calculate them.

If we used Cowan's equation (T_{UC}) the predicted results were always higher then the real ones. On the other hand the skew-bending theory, (T_{US}), gave a lower value every time. In the last column we can see the ultimate load for the concrete struts according to the Hungarian Code. We can observe that this value is always much higher than any of the others. This means that the failure should have occurred due to the yielding of the reinforcement. This was proved by the tests: the cause of the failure for the reinforced beams was the yielding of the reinforcement. One should also keep in mind that the lowest value of the calculated three is the ultimate so the codes were conservative in all the cases (the value for concrete failure was even higher for EC2).

	Measured	Average	T_{UR1}	T_{UR2}	T_{UC}	T_{US}	T_{UH3}
B-1	2.00	1.88	1.20	1.20	2.40	**1.34**	6.12
B-2	2.00				2.46	**1.38**	6.51
B-3	1.64				2.39	**1.33**	6.03
C-1	2.54	2.33	2.39	1.20	3.38	**1.36**	6.28
C-2	2.25				3.38	**1.36**	6.28
C-3	2.21				3.32	**1.31**	5.87
D-1	1.79	2.03	1.20	2.39	2.25	**1.23**	5.16
D-2	2.25				2.45	**1.37**	6.43
D-3	2.06				2.43	**1.36**	6.28
E-1	3.38	3.15	2.39	2.39	3.47	**1.41**	6.82
E-2	2.85				3.40	**1.36**	6.35
E-3	3.23				3.43	**1.40**	6.59

Table 4. The measured and the calculated ultimate moments (kNm)

5.3 Angle of cracks

As previously mentioned the angle of the crack appears in the equations of the Eurocode2. A similar equation can be obtained from Rausch's model by equating the ultimate torsional moments calculated for the failure of the longitudinal bars and for the stirrups. The two different equations are:

$$tg^2\alpha = \frac{A_{sw}/s}{\sum A_{sl}/u_k} \quad \text{(EC2)} \qquad\qquad tg\alpha = \frac{A_t/s}{\sum A_l/u_k} \quad \text{(Rausch's)}$$

Both equations predict a reduced slope for the cracks if the longitudinal reinforcement ratio is higher than the transversal (D) and the opposite if the stirrup spacing is smaller (C). Furthermore according to the theory of Saint-Venant the angle of the first crack is exactly 45° since at that stage we neglect the effects of the reinforcing. This angle should change later showing the influence of the reinforcement ratio in the two directions. As the angle changes the crack should rotate somehow with the increasing of the load. The results of the tests confirmed only the first part of this assumption. This can be seen in Table 5., the angle of the cracks were close to 45° in every case. Unfortunately due to the incoherent crack propagation, the small dimensions and limited number of the specimens, no proof could be obtained for the second theoretical assumption.

	Measured	EC2	Rausch
B	54,4	45,0	45,0
C	50,1	54,7	63,4
D	51,4	35,2	26,6
E	47,6	45,0	45,0

Table 5. The angle of the cracks (degrees)

6. CONCLUSIONS

At the beginning of this paper a brief review of the methods for calculating the torsion of reinforced concrete was presented. Later the results of the different equations for cracking and ultimate moment were compared with the tests carried out at the Technical University of Budapest.

The results showed good agreement with the theoretical values in case of the cracking moment and showed that both Eurocode2 and the Hungarian Code were conservative in predicting the ultimate load. Additionally the experiments could not verify the assumption that the angle of the crack at failure highly depends on the reinforcement. In the future we are planning to carry out more experiments on fiber reinforced concrete elements.

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